Int J Theor Phys (2007) 46: 2283–2298
DOI 10.1007/s10773-007-9346-x

Magnetohydrodynamics and Plasma Cosmology

Kostas Kleidis · Apostolos Kuiroukidis · Demetrios Papadopoulos · Loukas Vlahos

Published online: 8 May 2007

Abstract We study the linear magnetohydrodynamic (MHD) equations, both in the Newtonian and the general-relativistic limit, as regards a viscous magnetized fluid of finite conductivity and discuss instability criteria. In addition, we explore the excitation of cosmological perturbations in anisotropic spacetimes, in the presence of an ambient magnetic field. Acoustic, electromagnetic (e/m) and fast-magnetosonic modes, propagating normal to the magnetic field, can be excited, resulting in several implications of cosmological significance.

1 Introduction

Although Plasma Physics and Cosmology are two well-established fields of Theoretical Physics, the formulation of magnetohydrodynamics in curved spacetime is a relatively new development [1]. In particular, in spite the fact that MHD processes in flat spacetime gained much attention [2], only recently we were able to discuss exact spherically symmetric MHD solutions within the context of General Relativity (GR) [3], something that gave rise to efforts of exploring MHD processes in the vicinity of central engines [4–6]. On the other hand, magnetic fields are known to have a widespread presence in our Universe, being a common property of the intergalactic medium in galaxy clusters [7], while, reports on Faraday rotation imply magnetic fields of significant strength at high redshifts [8, 9]. These results lead the scientists to go even further and to look for gravitational instabilities in magnetized cosmological spacetimes, either in the Newtonian [10] or the GR limit [11–16].

K. Kleidis (✉) · A. Kuiroukidis · D. Papadopoulos · L. Vlahos
Department of Physics, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece
e-mail: kleidis@astro.auth.gr

K. Kleidis
Department of Civil Engineering, Technological Education Institute of Serres, 62124 Serres, Greece

A. Kuiroukidis
Department of Informatics, Technological Education Institute of Serres, 62124 Serres, Greece

In the present article, we intend to re-construct the MHD equations, both in the Newtonian and the GR limit and to use them in order to study the finite-amplitude wave propagation in MHD media. The reason to do so, actually relies on the thermal history of the Universe.

According to the Standard Model [17, 18], after nucleosynthesis the Universe goes on expanding and cooling but nothing of great interest takes place until $t \sim 10^{13}$ s ($z \sim 10^3$). At that time, the temperature drops to the point where electrons and nuclei can form stable atoms (recombination epoch). During the so-called *radiation epoch*, photons couple strongly with matter, the main constituent of which is in the form of *plasma*. Interactions between the various constituents of the Universal matter content along this period include radiation-plasma coupling which is described by plasma dynamics. On the other hand, the presence of plasma plays an important role in shaping the radiation spectrum, something that is fortified by the fact that there appear to exist cosmic magnetic fields, of the order 10^{-6} G, which evidently have not had enough time to evolve during the gravitating epoch of the Universe [7, 12]. Therefore, although it is not traditional to characterize the radiation epoch by the dominance of plasma interactions, it may well be also called the *plasma epoch*. It is the time-period during which the electromagnetic interaction dominates among the four fundamental forces.

2 MHD Equations in an Expanding Universe

2.1 The Newtonian MHD Theory

During the plasma epoch, the MHD equations in the Newtonian theory of gravity are given by [19, 20]

$$\rho\left(\frac{Du_i}{dt}\right) = -\frac{\partial p}{\partial x_i} + \rho\frac{\partial U}{\partial x_i} + \frac{1}{4\pi}\frac{\partial(H_i H_j - \frac{1}{2}\delta_{ij}H^2)}{\partial x_j}, \tag{1}$$

$$\frac{1}{\rho}\frac{D\rho}{dt} = -\frac{\partial u_i}{\partial x_i}, \tag{2}$$

$$\nabla^2 U = -4\pi G\rho, \tag{3}$$

$$\frac{\partial H_i}{\partial t} = \frac{\partial(u_i H_j - u_j H_i)}{\partial x_j}, \qquad \frac{\partial H_i}{\partial x_i} = 0. \tag{4}$$

In (1–4), Latin indices refer to the four-dimensional spacetime (in accordance Greek indices refer to the three-dimensional spatial section), δ_{ij} is the Kronecker symbol and we have used the notation

$$\frac{D}{dt} = \frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_j}.$$

On the other hand, ρ and p denote the energy-density and the pressure of a magnetized perfect fluid, H_i are the strength-components of the ambient magnetic field and U is the gravitational potential. Introducing the scalars [21]

$$2\sigma^2 = \sigma_{ij}\sigma_{ij}, \qquad 2\omega^2 = \omega_{ij}\omega_{ij}, \tag{5}$$

$$2\tilde{\sigma}^2 = \tilde{\sigma}_{ij}\tilde{\sigma}_{ij}, \qquad 2\tilde{\omega}^2 = \tilde{\omega}_{ij}\tilde{\omega}_{ij} \tag{6}$$

where

$$\sigma_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\theta\right),\qquad \omega_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right),$$
$$\tilde{\sigma}_{ij}=\frac{1}{2}\left(\frac{\partial H_i}{\partial x_j}+\frac{\partial H_j}{\partial x_i}\right),\qquad \tilde{\omega}_{ij}=\frac{1}{2}\left(\frac{\partial H_i}{\partial x_j}-\frac{\partial H_j}{\partial x_i}\right) \tag{7}$$

and $\theta=\frac{\partial u_i}{\partial x_i}$ is the *expansion* parameter, we obtain

$$u_{i,j}=\sigma_{ij}+\omega_{ij}+\frac{\theta}{3}\delta_{ij}. \tag{8}$$

Combining (1–4) with (8), we find

$$\frac{1}{\rho}\frac{D^2\rho}{dt^2}=\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial p^*}{\partial x_i}\right]+4\pi G\rho+\frac{H_j}{4\pi\rho^2}\frac{\partial H_i}{\partial x_j}\frac{\partial\rho}{\partial x_i}$$
$$+2\left[\frac{2\theta^2}{3}+\left(\sigma^2-\frac{\tilde{\sigma}^2}{4\pi\rho}\right)-\left(\omega^2-\frac{\tilde{\omega}^2}{4\pi\rho}\right)\right] \tag{9}$$

where, we have set $p^*=p+\frac{H^2}{8\pi}$ to denote the *generalized pressure* involved, due to the perfect fluid plus the magnetic field. Equation (9) governs the finite-amplitude wave propagation in Newtonian MHD theory. When $H_i=0$, it reduces to the corresponding law derived by Hunter [22]. On the other hand, using the notation

$$\frac{DH_i}{dt}=\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial t}\right)H_i=H_j\frac{\partial}{\partial x_j}-H_i\frac{\partial}{\partial x_j} \tag{10}$$

together with (8), we obtain

$$\frac{DH_i}{dt}=\left(\sigma_{ij}+\omega_{ij}-\frac{2}{3}\theta\delta_{ij}\right)H_j \tag{11}$$

and multiplying (11) by μH_i, we end up an equation for the temporal evolution of the magnetic field's energy density, namely

$$\frac{D}{dt}\left(\frac{\mu H^2}{8\pi}\right)=\frac{\mu}{4\pi}\sigma_{ij}H_iH_j-\frac{4\theta}{3}\left(\frac{\mu H^2}{8\pi}\right) \tag{12}$$

where, μ is the permeability [11].

Summarizing, the MHD dynamics in the Newtonian limit is determined by the following system

$$\rho\left(\frac{Du_i}{dt}\right)=-\frac{\partial p}{\partial x_i}+\rho\frac{\partial U}{\partial x_i}+\frac{1}{4\pi}\frac{\partial(H_iH_j-\frac{1}{2}\delta_{ij}H^2)}{\partial x_j}, \tag{13}$$

$$\frac{1}{\rho}\frac{D^2\rho}{dt^2}=\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}\frac{\partial p^*}{\partial x_i}\right]+4\pi G\rho+\frac{H_j}{4\pi\rho^2}\frac{\partial H_i}{\partial x_j}\frac{\partial\rho}{\partial x_i}$$
$$+2\left[\frac{2\theta^2}{3}+\left(\sigma^2-\frac{\tilde{\sigma}^2}{4\pi\rho}\right)-\left(\omega^2-\frac{\tilde{\omega}^2}{4\pi\rho}\right)\right], \tag{14}$$

$$\nabla^2 U = -4\pi G\rho, \tag{15}$$

$$\frac{D}{dt}\left(\frac{\mu H^2}{8\pi}\right) = \frac{\mu}{4\pi}\sigma_{ij}H_i H_j - \frac{4\theta}{3}\left(\frac{\mu H^2}{8\pi}\right). \tag{16}$$

2.2 The Relativistic MHD Theory

To derive the general relativistic MHD equations, we start up with the Einstein field equations [19]

$$\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R} = \kappa T_{ab}, \quad \kappa = -\frac{8\pi G}{c^4} \tag{17}$$

where, $\mathcal{R}_{ab}$ is the Ricci tensor, $\mathcal{R} = g_{ab}\mathcal{R}^{ab}$ is the corresponding scalar curvature and T_{ab} is the energy-momentum tensor, representing the average state of the matter-energy content. Equation (17), together with the Bianchi identities

$$\left(\mathcal{R}^{ab} - \frac{1}{2}g^{ab}\mathcal{R}\right)_{;ab} = 0, \tag{18}$$

result in the conservation law $T^{ab}_{;b} = 0$, where, the semicolon denotes covariant derivative. In what follows, we perform our calculations in the system of geometrical units, where $G = c = 1$, admitting that the fluid's four-velocity satisfies the condition $u_a u^a = -1$; i.e. we place ourselves in a coordinate system comoving with the fluid.

Again, we consider a magnetized perfect fluid source, determined by the following energy-momentum tensor

$$T^{ab} = \left(\rho + \frac{H^2}{2}\right)u^a u^b + \left(p + \frac{H^2}{2}\right)h^{ab} - H^a H^b \tag{19}$$

where,

$$h^{ab} = g^{ab} + u^a u^b$$

is the projection tensor and

$$\epsilon = \rho + \rho\Pi$$

is the *total energy density*, due to the mass-energy content and the internal motions $(\rho\Pi)$. By analogy to the corresponding evaluation in the Newtonian limit, we introduce the *expansion* $\theta = u^a_{;a}$, the *shear* $\sigma_{ab} = h^c_a h^d_b u_{(c;d)} - \frac{1}{3}\theta h_{ab}$ and the *twist* $\omega_{ab} = h^c_a h^d_b u_{[c;d]}$ (round brackets denote symmetrization while square brackets antisymmetrization), to write the covariant derivative of the fluid's four-velocity in the form

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\theta h_{ab} - \dot{u}_a u_b \tag{20}$$

where, we have set $\dot{u}^a = u^a_{;c}u^c$.

Accordingly, the MHD dynamics in the GR limit is determined by the following system [11]

$$x\dot{u}^a + \dot{x}u^a + x\theta u^a + \left(p + \frac{H^2}{2}\right)_{;b} g^{ab} = (H^a H^b)_{;b}, \tag{21}$$

$$\begin{aligned}\left(\rho - \frac{H^2}{2}\right)_{;ab} u^a u^b = h^{ab}\left(p + \frac{H^2}{2}\right)_{;ab} + 2\frac{d}{dt}(H^2\theta) \\ - (H^a H^b)_{;ab} + 2x\left(\frac{2\theta^2}{3} + \sigma^2 - \omega^2 - \dot{u}^a\dot{u}_a\right) \\ + \frac{x}{2}(\rho + 3p + H^2) + 2\dot{u}_a (H^a H^b)_{;b} + (H^2)_{;a}\dot{u}^a\end{aligned} \tag{22}$$

and

$$\frac{\mu\dot{H}^2}{8\pi} = \frac{\mu}{4\pi}\sigma_{ij} H^i H^j - \frac{4\theta}{3}\left(\frac{\mu H^2}{8\pi}\right) \tag{23}$$

where, we have set $x = \rho + p + H^2$. Equations (21–23) may be applied to the investigation of perturbation effects in curved spacetime and, hence, to the discussion of linearized stability criteria. In order to obtain the perturbed expression of the above equations, we follow the well-established method imposed by Hawking [23].

3 Part I: MHD Phenomena along the Plasma Epoch

To discuss MHD phenomena in the early stages of the Universe, we use a solution to the Einstein–Maxwell equations, first developed by Thorne [24] and later discussed by Jacobs [25], which contains an ambient magnetic field. The model is homogeneous and has two equivalent "transverse" directions and one inequivalent "longitudinal" direction at each point of the spacetime. The magnetic field is frozen into the matter content and is directed along the longitudinal direction ($\hat{z}$-direction). We consider that the matter-content filling this model is in the form of the perfect fluid (19), which obeys the equation of state $p = \gamma\rho$, with $1/3 \leq \gamma \leq 1$. In this case, the line-element reads [24]

$$ds^2 = -dt^2 + t(dx^2 + dy^2) + t^m dz^2 \tag{24}$$

where, $m = 2\frac{(1-\gamma)}{(1+\gamma)}$. With the aid of (24), we may apply the GR limit of MHD dynamics to discuss stability of Bianchi-Type I cosmological models. In this case, we depart from the traditional approach, adopting an anisotropic cosmological model, in which the inherent magnetic field is responsible for the initial anisotropy of the Universe.

3.1 Excitation of Low-Frequency Plasma Waves

To deal with small-amplitude waves, we first assume a background situation, representing a uniform plasma in curved spacetime, in the presence of an ambient magnetic field. In our case, this situation is described by the solution (24), together with the energy-momentum tensor (19). Accordingly, we introduce first-order perturbations to the MHD equations (21–23) and taking into account the so-called Cowling approach [26], which admits that $\delta g_{ab} = 0$, we neglect all terms higher or equal than the second order. Eventually, we search for solutions to the linearized MHD equations in which all the perturbation quantities are written in the form of a plane-wave, with angular frequency n and a wave-vector

$\vec{k} = (0, 0, k^3)$ parallel to the magnetic field, $\vec{H} = (0, 0, H^3)$. Accordingly,

$$\delta\epsilon, \delta p, \delta H^a, \delta u^a \sim \exp i(nt - kz). \tag{25}$$

Upon consideration of the perturbed MHD equations [23], after straightforward but tedious calculations, we obtain the equation governing the density fluctuations [11]

$$\begin{aligned}\frac{\partial^2 \delta\rho}{\partial t^2} + A_1 \frac{\partial \delta\rho}{\partial t} - \delta p_{;ab} h^{ab} - A_2 c_s^2 \frac{\partial^2 \delta\rho}{\partial z^2} - [A_3 + c_s^2 A_4]\delta\rho \\ = c_1 \delta u^0 + c_3 \delta H^3_{;3} + F_1 \delta H^3 + F_2 \delta H^3_{;ab} + F_3 \delta H^3_{;b3}\end{aligned} \tag{26}$$

where, $c_s^2 = \delta p/\delta\rho$ is the speed of sound and A_i, c_i, F_i $(i = 1, 2, 3)$, are all functions of t and γ, the explicit form of which is given in [11]. Inserting (25) into (26), we obtain a secular equation of the form

$$n^2 - inA - S_1 - ikS_2 = 0 \tag{27}$$

where, A, S_1 and S_2 are complicated expressions of the background energy density and the pressure, as well as of the shear and the expansion of the Universe, together with the magnetic field's strength (see Appendix 1). Equation (27) results in

$$n_1 = \tau \cos\omega/2 + i\left[\frac{A}{2} + \tau \sin\omega/2\right] \tag{28}$$

and

$$n_2 = -\tau \cos\omega/2 + i\left[\frac{A}{2} - \tau \sin\omega/2\right] \tag{29}$$

where, we have set

$$\tau = 4\sqrt{\left(S_1 - \frac{A^2}{4}\right)^2 + k^2 S_2^2} \tag{30}$$

and

$$\cos\frac{\omega}{2} = \frac{kS_2}{\tau}, \qquad \sin\frac{\omega}{2} = \frac{4S_1 - A^2}{4\tau}. \tag{31}$$

According to (28) and (29), the medium remains stable when the frequency equals to n_1. In this case, the magnetic field's perturbation, δH^3, turns out to be an *Alfven wave*. On the other hand, for $n = n_2$, the medium becomes unstable and δH^3 grows in time. We identify this instability with the well known *Parker's instability* [27].

There are also other potential instabilities that can be traced within the context of general relativistic MHD. In particular, a spectrum of magnetized sound waves may be excited and form large-scale damped oscillations in the expanding Universe (e.g. see [16]). The characteristic frequency of the excited waves is slightly shifted away from the sound frequency and the shift depends on the strength of the primordial magnetic field. This magnetic-field-dependent shift, may have an effect on the acoustic peaks of the CMRB.

One of the main scopes of this study, is to examine the spectrum of the unstable low frequency plasma waves in the anisotropic cosmological model (24). Under the assumption that the magnetic field lays along the $\hat{z}$-direction and the perturbations are plane-waves propagating also in the $\hat{z}$-direction, the perturbed general relativistic MHD equations, result in the *dispersion relation*

$$
\begin{aligned}
&- n^2 + k^2 c_s^2 - (1 + c_s^2)\bigg[-\frac{8}{3}\theta^2 + 2\sigma^2 + \frac{1}{2}(\rho + 3p) \\
&\quad + \frac{2\theta H_{,0}^2}{\rho + p} + \frac{4\theta^2 H^2}{3(\rho + p)} - \frac{2H^2}{3}\bigg] - \frac{1}{2}(p + \rho)(1 + 3c_s^2) \\
&\quad - in\bigg[4\theta + \frac{2H_{,0}^2}{(\rho + p)} - \frac{4\theta H^2}{3(\rho + p)}\bigg] \\
&= (\mathcal{R}_1 + in\mathcal{R}_2)\frac{\delta H^3}{\delta\epsilon}
\end{aligned} \tag{32}
$$

where, we have set

$$
\frac{\delta H^3}{\delta\epsilon} = \frac{H^3}{4\pi\mathcal{D}(2-\gamma)}[1 - \Lambda_1 - in\Lambda_2], \tag{33}
$$

and $\mathcal{R}_1$, $\mathcal{R}_2$, Λ_1, Λ_2 and $\mathcal{D}$, are functions of the magnetic strength H, the shear σ and the expansion θ of the background solution (see Appendix 2). At the early stages of the evolution ($t \to 0$), (32) reads

$$
-n^2\bigg[1 - \frac{u_A^2}{2-\gamma}\bigg] + k^2 c_s^2 + \frac{k^2 u_A^2}{2-\gamma} + T_1(t,\gamma) + inT_2(t,\gamma) = 0 \tag{34}
$$

where, u_A is the Alfven velocity and $T_1(\gamma,t)$, $T_2(\gamma,t)$ are complicated expressions of γ and t, the exact form of which is given in Appendix 2. We re-write (34) in the form $D_r + iD_i = 0$ and, assuming that $n = n_r + in_i$, where, the real part of the excited frequency is much larger than the corresponding imaginary part ($n_r \gg n_i$), we obtain:

- The real part of the excited frequency from the equation $D_r = 0$

$$
n_r^2\bigg[1 - \frac{u_A^2}{2-\gamma}\bigg] = k^2 c_s^2 + \frac{k^2 u_A^2}{2-\gamma} + \frac{\tilde{T}_1(\gamma)}{t^2}. \tag{35}
$$

- The imaginary part of the excited frequency from the relation [28]

$$
n_i = -\frac{D_i(n_r,k)}{\frac{\partial D_r(n_r,k)}{\partial n}|_{n=n_r}} \tag{36}
$$

which yields

$$
\begin{aligned}
n_i &= \frac{T_2(\gamma,t)}{2(1 - u_A^2(t)/(2-\gamma))} \\
&= \bigg(\frac{C(\gamma)}{t}\bigg)\bigg(\frac{1}{2(1 - u_A^2(t)/(2-\gamma))}\bigg).
\end{aligned} \tag{37}
$$

For $t \to 0$, n_i becomes negative, decaying rapidly at late times. Therefore, we verify that the role of magnetic field is to make the expanding Universe even more stable against the expected *Jeans-type instabilities*.

On the other hand, the real part of the frequency is shifted away from the frequency of sound

$$n_r^2 = \frac{k^2 c_s^2 + \frac{k^2 u_A^2}{2-\gamma} + \frac{\tilde{T}_1}{t^2}}{1 - \frac{u_A^2(t)}{2-\gamma}}. \tag{38}$$

In fact, as $\gamma \to 1/3$, the magnetic field vanishes and the anisotropic model (24) approaches the weakly-magnetized Friedmann–Robertson–Walker (FRW) model, which is used extensively in the literature (e.g. see [31] and references therein). In addition, the characteristic frequency n_r approaches the value $n_r = kc_s + \Delta n_r(u_A)$. The presence of the Alfven velocity in (38), may be responsible for the distortion of acoustic peaks, as it was pointed out first by [12]. Therefore, a spectrum of low frequencies will be excited with frequency n_r.

3.2 Evolution of the Density Fluctuations

Finally, we examine the temporal evolution of the density fluctuations. A linear density perturbation has the form

$$\delta\rho \sim [\rho_0 e^{|B(\gamma,t)|}] e^{i(n_r t - kz)} \tag{39}$$

where,

$$B(\gamma, t) = \frac{C(\gamma)}{2^{\frac{1-u_A^2(t)}{2-\gamma}}}.$$

The function $|B(\gamma, t)|$ decays as time grows or, equivalently, as γ decreases. In particular, as $t \to 0$, which corresponds to $\gamma \to 1$, we have $u_A \to 1$ and $|B(\gamma, t)| \to \infty$. On the other hand, for $t \to \infty$, which results in $\gamma \to 1/3$, we obtain $u_A \to 0$ and $|B(1/3, t)| \to (3/2)$. These results indicate that, at early times ($t \to 0$), the perturbations' amplitude becomes large and the excited waves will form a *spectrum of damped oscillations* within the magnetized cosmological model. We may search for the source of this damping by analyzing the functional form of $T_2(\gamma, t)$ in (34). In the absence of magnetic fields ($\gamma = 1/3$), we obtain $T_2 = -4\theta$ and the expanding plasma stabilizes the ion-acoustic waves.

On the other hand, if the cosmological model is *static* ($A = const.$ and $\theta \sim \frac{\dot{A}}{A} = 0$), stabilization of the magnetosonic waves can be caused by the magnetic pressure. In a magnetized expanding Universe (see (24)) both processes will be combined to damp the large-amplitude oscillations. Assuming, for the sake of agreement, that at a some time $t = const.$ the Universe becomes static, a *Jeans-like instability* sets in (see also [29]). Therefore, we realize that the damping is caused by the *free energy*, available both in the plasma and the magnetic field, during the early stages of the Universal expansion.

Summarizing, we have analyzed the small-amplitude wave propagation in the early Universe, using an anisotropic cosmological model with an *inherent* magnetic field towards the $\hat{z}-$direction, which is frozen into a perfect fluid with equation of state $p = \gamma\rho$ (where, $1/3 \le \gamma \le 1$). For $\gamma \to 1/3$, the anisotropic model used, approaches the weakly magnetized FRW model, in which the magnetic field is a small linear perturbation (e.g. see [14, 30]).

4 Part II: MHD Phenomena of Cosmological Significance

Newtonian Cosmology can be a quite good approximation for gravity, at scales much smaller than the Hubble length. For this reason, it is often used to describe our Universal neighborhood ($0.1 \le z \le 1.5$), within the context of Observational Cosmlogy; in particular, as regards the local observations of cosmological significance [18]. Within the context of Newtonian Cosmology, we study the linear MHD behavior of a viscous magnetized fluid with finite *conductivity* and generalize the *Jeans instability criterion*. In particular, we look into the linear evolution of small inhomogeneities in the cosmic medium and examine how the magnetic field and the fluid's viscosity may affect the characteristic scales of the gravitational instabilities. Accordingly, we discuss the electrodynamic properties of the collapsing fluid, the resulting amplification of the magnetic field and the formation of unstable *current-sheets*.

A central point to our discussion is the concept of *anomalous resistivity*, which is triggered by electrostatic instabilities in the plasma and may reduce substantially the corresponding electrical conductivity. We argue that these changes in the resistivity of the protogalactic medium might have lead to the formation of strong electric fields during galactic collapse. These fields can accelerate the abundant free electrons and ions, to produce Ultra High Energy Cosmic Rays (UHECRs) during the formation of protogalactic structures.

In Newtonian Cosmology, one deals with the following set of MHD equations [19]

$$\frac{\partial\rho}{\partial t} = -3\frac{\dot{a}}{a}\rho - \frac{1}{a}\vec{\nabla}\cdot(\rho\vec{u}), \tag{40}$$

$$\frac{\partial\vec{u}}{\partial t} = -\frac{\dot{a}}{a}\vec{u} - \frac{1}{a}(\vec{u}\cdot\vec{\nabla})\cdot\vec{u} - \frac{c_s^2}{a\rho}\vec{\nabla}\rho + \frac{1}{a}\vec{\nabla}\phi$$
$$+ \tfrac{1}{4\pi a\rho}(\vec{\nabla}\times\vec{B})\times\vec{B} + \tfrac{\nu}{a^2\rho}\nabla^2\vec{u}, \tag{41}$$

$$\nabla^2\phi = -4\pi G a^2\rho, \tag{42}$$

$$\frac{\partial\vec{B}}{\partial t} = -2\frac{\dot{a}}{a}\vec{B} + \frac{1}{a}\vec{\nabla}\times(\vec{u}\times\vec{B}) + \frac{\eta}{a^2}\nabla^2\vec{B}, \tag{43}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{44}$$

where, $a(t)$ is the *radius* of the Newtonian Universe and ϕ, stands for the gravitational potential.

4.1 Magnetically Induced Anisotropies in the Gravitational Collapse

Once again, we perturb (40–44) around a zeroth order background solution, representing a uniform plasma in the presence of an ambient magnetic field

$$\rho = \rho_0 + \rho_1, \qquad \vec{B} = \vec{B}_0 + \vec{B}_1, \qquad \phi = \phi_0 + \phi_1, \qquad \vec{u} \neq 0.$$

We assume plane-wave perturbations and take the time-derivative of (41). The real part yields

$$\ddot{\vec{u}} = -\left(H + \frac{\nu k^2}{a^2\rho_0}\right)\dot{\vec{u}} + \left[8\pi G\rho_0 - \frac{k^2}{a^2}\left(c_s^2 + u_A^2 + \frac{\nu H}{\rho_0}\right)\right]\vec{u}. \tag{45}$$

Furthermore, we assume that the perturbations propagate (a) parallel to the magnetic field and (b) perpendicular to the magnetic field. Accordingly, we end up with two *critical wavelengths*, associated to the corresponding Jeans scales, namely

$$\lambda_\perp \simeq \sqrt{\frac{c_s^2 + u_A^2 + \nu H/\rho_0}{8\pi G\rho_0}}, \qquad \lambda_\parallel \simeq \sqrt{\frac{c_s^2 + \nu H/\rho_0}{8\pi G\rho_0}} \tag{46}$$

where, $H = \frac{\dot{a}}{a}$ is the Hubble parameter, ν is the viscosity of the plasma fluid and η is the corresponding resistivity. From (46), it is evident that the magnetic field *induces* a degree of anisotropy in the gravitational collapse and, therefore, one expects gradual formation of turbulent motions within the magnetized medium. These motions, in turn, contribute additional terms to the viscosity of the fluid. In the post-recombination epoch, the viscosity due to turbulent motions reads

$$\nu_{\rm turb} \sim \rho_0 u_1 l_{\rm mix} \tag{47}$$

where, $l_{\rm mix}$ the turbulent mixing length. Assuming that

- u_1 reaches values close to u_A
- $l_{\rm mix} \ll \lambda_\perp$ and
- $T_{\rm rec}$ is known

we end up with the condition

$$c_s^2 < \frac{\nu H}{\rho_0} < u_A^2. \tag{48}$$

At the present epoch, we adopt the typical values [18]: For the magnetic field, $B \leq 10^{-7}$ G; for the energy-density, $\rho_0 \geq 10^{-29}$ gr cm^{-3} and for the Hubble parameter, $H = 100h$ km s^{-1} Mpc^{-1}, where $0.4 \leq h \leq 1$, we find

$$\frac{\lambda_\perp}{\lambda_\parallel} \sim \sqrt{\frac{c_{\rm a}^2 \rho_0}{\nu_{\rm turb} H}} \gg 1 \tag{49}$$

with $\lambda_\perp$ being of the order of a (comoving) Mpc. Following the standard structure formation scenarios, the initial collapse of a large structure will be followed by successive fragmentation into smaller scale formations with characteristic lengths $\lambda \ll \lambda_\perp$. Moreover, as we will outline next, the magnetic field which is trapped into the gravitating medium, will be increased even further, due to the anisotropic collapse. In the case of an almost spherically symmetric collapse, linear inhomogeneities in the magnetic field's energy density amplify in tune with those in the density of the matter, so that

$$\delta B^2 \propto \delta\rho$$

where, $\delta B^2 = B_1^2/B_0^2$ and $\delta\rho = \rho_1/\rho_0$. Therefore, even in the spherically symmetric case, the formation of matter condensations in the post-recombination Universe signals the amplification of any magnetic field that happens to be present at that time [32].

We have already seen that the anisotropic magnetic field induces some degree of anisotropy in the gravitational collapse of the protogalactic structure. On the other hand, this magnetically induced anisotropy will react back to affect the evolution of the field itself. The evolution of the magnetic field during the nonlinear regime of a generic, non-spherical protogalactic collapse, has been considered by a number of authors (e.g. see [31] and references there in). The approach is both analytic and numerical and agree that *shear effects* increase the final strength of the magnetic field, while confining it onto the protogalactic plane. Compared to the magnetic strength of the spherical collapse scenario, this anisotropic increase is larger by, at least, one order of magnitude and, therefore, protogalactic structures can be endowed with magnetic fields considerably stronger than those previously anticipated.

So far, we have seen how the magnetic presence modifies the way the gravitational collapse may proceed, by changing the overall stability of the magnetized fluid. This, in turn, affects the evolution of the magnetic field itself and may trigger a chain of nonlinear effects on certain scales.

At next, we will see that, selective amplification of certain perturbation modes, plays an important role during the nonlinear stages of protogalactic collapse, helping instability to reach its *saturation point*.

4.2 Amplification of the Induced Electric Currents

In the presence of magnetic fields there is an electric current induced by $\vec{B}$, namely

$$\vec{J} = \frac{c}{4\pi}\vec{\nabla} \times \vec{B} = \frac{c}{4\pi}\alpha\vec{B} \tag{50}$$

where, $\vec{\nabla} \times \vec{B} = \alpha\vec{B}$ and α measures the magnetic torsion (see [33] for details). One expects that, initially, α is small. However, the subsequent fragmentation of the protogalactic cloud to scales with $\lambda \ll \lambda_\perp$ will increase $\vec{\nabla} \times \vec{B}$ and strengthen the induced current. In particular, for $B \sim 10^{-7}$ G, we obtain $J \sim 10^2\alpha$ which may reach appreciable strengths for reasonable values of α.

When the induced current exceeds a critical value, namely $J_c \sim \rho(e/m_p)c_s$, where, $c_s \sim 10^4\sqrt{T(\mathrm{K})}$ cm/s, m_p is the ion mass and e is the electron charge, the excitation of low frequency electrostatic turbulence will increase the resistivity of the medium by several orders of magnitude [34]. In fact, even small values of α may lead to $J > J_c$, thus making the plasma electrostatically unstable. This effect, which is known as *anomalous resistivity*, can be explained by the development of current-driven electrostatic instabilities in plasma, that could lead to the excitation of a variety of waves and oscillations. Absorption of these waves by the ions, results in a transfer of momentum from the electrons to the ions, along with the usual momentum loss from the former species to the latter. The average momentum loss by the electron per unit time can be written as an effective collision term, in the form

$$nm_e\nu_{\mathrm{eff}}\vec{u}_e = -\vec{F}_{\mathrm{fr}} \tag{51}$$

where, F_{fr} is the average *friction force* and $n = \rho/m_p$ is the ambient number density of the plasma particles. The friction force is proportional to both the linear growth rate of the electrostatic waves (γ_k) and the energy of the excited waves (W_k). The effective collision frequency is estimated to be $\nu_{\mathrm{eff}} = \omega_e(W_{\mathrm{sat}}/k_BT)$. Then, the anomalous resistivity will be

$$\eta_{\mathrm{an}} \sim \frac{\nu_{\mathrm{eff}}c^2}{4\pi\omega_e^2} \sim \left(\frac{W_{\mathrm{sat}}}{k_BT}\right)\left(\frac{c^2}{4\pi}\right)\frac{1}{\omega_e} \tag{52}$$

where, $\omega_e = 5.6 \times 10^4\sqrt{n}$ s^{-1} is the plasma frequency, k_B is the Boltzman constant and $(W_{\mathrm{sat}}/k_BT) \sim 10^{-6}$–$10^{-4}$ is the saturated level of the electrostatic waves [34]. For certain types of current-driven waves, the anomalous resistivity is several orders of magnitude above the classical one, as confirmed in numerous laboratory experiments [35].

This sudden switch to high electrical resistivity leads to the formation of strong electric currents and therefore to a fast magnetic dissipation and intense plasma heating. The electric fields induced by the gravitational collapse will be

$$E \sim \frac{u_A}{c}B + \eta_{\mathrm{an}}J_c \sim \eta_{\mathrm{an}}J_c \tag{53}$$

provided that $\frac{u_A}{c} \ll 1$. According to this scenario, the gravitational collapse of a magnetized dust cloud amplifies the magnetic field and (indirectly) generates strong electric currents, which form large-scale current-sheets localized on the protogalactic plane.

The anisotropy of the collapse enhances the local currents even further and, eventually, drives the resistivity towards anomalously high values. The inevitable result is strong electric fields accelerating the abundant free electrons. In particular, the energy gained by an electron traveling a length, $\lambda \sim 10^{-3}\lambda_\perp \ll \lambda_\perp$ is $W_{\rm kin} \sim eE\lambda$ and the relativistic factor, $\gamma = [1 - (v/c)^2]^{-1}$ is given by

$$\gamma = \frac{W_{\rm kin}}{m_e c^2} \sim \frac{eE\lambda_\perp}{10^3 m_e c^2} \sim \frac{10 e^2 \eta_{\rm an}\sqrt{T}B}{m_e m_p c^2 G^{1/2}} \quad \Rightarrow$$

$$\gamma \sim 10^{14}\left(\frac{n}{10^{-4}\ \mathrm{cm}^{-3}}\right)^{-\frac{1}{2}}\left(\frac{T}{1\ \mathrm{K}}\right)^{\frac{1}{2}}\left(\frac{B}{10^{-7}\ \mathrm{G}}\right) \tag{54}$$

in CGS units. Taking into account that

$$\lambda_\perp \sim B/nm_p\sqrt{G},$$

$$c_{\rm s}^2 < \nu H/\rho < u_A^2$$

and, through (52),

$$W_{\rm sat}/k_B T \sim 10^{-6},$$

we find that the typical energy gain by a free electron may reach extremely high values within short timescales ($t_{\rm acc} \sim \lambda/c \ll \lambda_\perp/c \sim 10^6$ yrs), even for relatively weak magnetic fields. It is straightforward to extend this process to proton acceleration and show that protogalactic collapse may be the source of UHECRs.

We also anticipate a few particles traveling across several fragments adding up to a scale comparable to the Jeans length. Many particles will drift in and out the primordial current-sheet, as well as the associated strong electric fields. Through synchrotron emission, particles gain and lose energy continuously. Provided that fragmentation has already taken place, these particles may diffuse along the different current-sheets, forming the observed power-law distribution. The role of unstable current-sheets on the acceleration of cosmic rays along the giant radio-galaxies, has already been pointed out in the literature [36].

Summarizing, the suggestion made here is that, in the context of Newtonian Cosmology, production of UHECRs may have started almost simultaneously with the formation of galaxies, through the electrodynamic characteristics of the gravitational instability. This process may be continued until today, since the previously described instability is active on all cosmic scales.

5 Conclusions

Part I:

1. The Jeans instability, established in an Einstein Universe with a very particular cosmological constant [29], is absent in the presence of magnetic fields, which act as stabilizing forces.
2. An anisotropic magnetized Universe re-enforces the stability, initially found in the FRW model, against Jeans-like instabilities.

3. A spectrum of low frequency and large amplitude damped oscillations may appear in the early Universe, with a characteristic frequency $n_r = kc_s + \Delta n$, where, Δn depends on the strength of the primordial magnetic field.

Because of the above results, we come to the following conclusions:

- The magnetic-field-dependent shift (Δn), may cause measurable distortion on the acoustic peaks of the CMRB (see also [12, 15, 30]).
- The amplitude of the excited waves is substantially large in the early stages of the Universal evolution and gradually decays.
- If the Universe has passed through a strongly magnetized anisotropic phase, the excited waves may be responsible for the formation of large-scale fluctuations, as it has been shown in 2-D numerical simulations [37].

Part II:

1. We have found that, a magnetic field frozen onto a fluid of finite viscosity, may alter the standard picture of the gravitational instability.
2. We have discussed how the preferential, anisotropic magnetic amplification may increase the induced electric currents on a plane perpendicular to the main axis of the collapse.
3. These gravitationally induced current-sheets will, in turn, trigger electrostatic instabilities, which may lead to anomalously high values of the resistivity and, consequently, to strong electric fields.
4. These electric fields can be strong enough, in order to accelerate free electrons up to ultra high energies, producing UHECRs.

To the best of our knowledge, this is the first time that a direct connection between gravitational instability and cosmic-ray acceleration has been suggested and discussed.

Acknowledgements The authors would like to thank Dr. C.G. Tsagas for helpful discussions. Financial support from the Greek Ministry of Education under the Pythagoras programm, is gratefully acknowledged.

Appendix 1

The expressions A, S_1 and S_2, appearing in (27), are written in the form:

$$A = \frac{1}{\rho + p}\left[\frac{13}{3}H^2\theta - (\rho + p)\theta + \frac{35}{3}H_{,z}H^z_{,0} + \frac{1}{2}g^{zz}g_{zz,0}H^2\frac{2\rho + 2p + H^2}{\rho + p + H^2}\right],$$

$$S_1 = -\frac{\rho}{2}(1 + c_s^2)(1 + 3c_s^2) + k^2\left[c_s^2 + \frac{H^2 - (\rho + p + 3H^2)H_z}{(\rho + p + H^2)}\right]$$

$$- c_s^2H^2 - \frac{1}{2}\left[\rho + 3p + \frac{4\theta^2}{3} + 4\sigma^2 + \frac{4}{3}H^2 - \frac{8}{3}H^zH_{z;0}\right.$$

$$+ \frac{2\theta}{\rho + p}\left(H_zH^z_{;0} + \frac{13}{3}\theta H^2 + \frac{32}{3}\theta H_zH^z_{;0}\right)$$

$$\left.+ \frac{1}{2}g^{zz}g_{zz;0}H_z(2\rho + 2p + H^2)(\rho + p + H^2)^{-1}\right](1 + c_s^2)$$

$$- \left\{H^z_{;00} - 6\theta H^z_{;0} - H^z\left[3H^2 + 6\sigma^2 - 6\theta^2 + \frac{\rho - 5p}{3}\right.\right.$$

$$
+\frac{1}{2}g^{zz}g_{zz,0}\left(\frac{1}{2}g^{zz}g_{zz,0}-\theta\right)\frac{2\rho+2p+H^2}{\rho+p+H^2}\bigg]-\frac{8}{3}H^2H^z_{;0}\bigg\}
$$

$$
-\bigg\{2H^z_{;0}H_{z;0}+H_zH^z_{;00}+\theta H_zH^z_{;0}+2H^2\left[2\sigma^2+\frac{\theta^2}{3}+\frac{1}{2}(\rho+p+H^2)\right]
$$

$$
+\frac{\rho_{,0}}{\rho+p}\left(H_zH^z_{;0}+\frac{13}{3}\theta H^2+\frac{32}{3}\theta H_zH^z_{;0}\right)+\frac{2H_zH^z_{;0}}{\rho+p+H^2}\left(p+\frac{H^2}{2}\right)_{;0}-\theta\rho_{;0}
$$

$$
+g^{zz}g_{zz,0}\bigg[5H^2\theta-H_zH^z_{;0}-H^2u^z_{;z}+\frac{H^2}{\rho+p+H^2}\left(p+\frac{H^2}{2}\right)_{;0}
$$

$$
-H^z\frac{2\rho+2p+H^2}{\rho+p+H^2}\left(\frac{H^z\rho_{,0}}{\rho+p}-H^z_{,0}\right)\bigg]\bigg\},
$$

$$
S_2=H_zH^z_{;0}+H^2u^z_{;z}-5H^2\theta-\frac{H^2}{\rho+p+H^2}\left(p+\frac{H^2}{2}\right)_{,0}+g^{zz}H^2\frac{2\rho+2p+H^2}{\rho+p+H^2}.
$$

Appendix 2

The functions $\mathcal{R}_1$, $\mathcal{R}_2$, Λ_1, Λ_2 and $\mathcal{D}$, appearing in (32) are:

$$
\mathcal{R}_1=[H_{3,00}-4(\theta H_{3,0}+\theta_{,0}H_3)-2H^3_{,0}\Gamma^0_{33}]+H_3\left[-\frac{4}{3}H^2+\frac{4}{3}\theta^2+2\sigma^2+2(\rho+2p)\right]
$$

$$
+g^{33}[-2\Gamma^0_{33,0}-2\Gamma^0_{33}\Gamma^b_{b0}+2\Gamma^0_{33}\Gamma^3_{30}]H_3(n^2+k^2)H_3,
$$

$$
\mathcal{R}_2=2H_{3,0}-4\theta H_3-\Gamma^0_{33}H_3g^{33},
$$

$$
\Lambda_1=\frac{1}{\mathcal{D}}[k^2(H^3)^2+(2\gamma-1)\theta[2x\Gamma^3_{30}+(p+H^2/2)_{,0}]-k^2c_s^2g^{33}(\epsilon+p)],
$$

$$
\Lambda_2=(2\gamma-1)x\theta,
$$

$$
\mathcal{D}=-n^2x+k^2(H^3)^2-(2-\gamma)\theta\left[2x\Gamma^3_{30}+\left(p+\frac{H^2}{2}\right)_{,0}\right]
$$

$$
-in\left[2x\Gamma^3_{30}+p+\frac{H^2}{2}-(2-\gamma)x\theta\right].
$$

Finally, the functions T_1 and T_2, appearing in (34), have the following functional form:

$$
T_1(\gamma,t)=\frac{\tilde{T}_1(\gamma)}{t^2},
$$

$$
\tilde{T}_1(\gamma)=-(9\gamma^6-993\gamma^5+4842\gamma^4-6818\gamma^3+2889\gamma^2
$$

$$
-461\gamma+788)/(6(\gamma-2)(\gamma-3)^2(1+\gamma)^2)
$$

and

$$T_2(\gamma,t) = -4\theta + \frac{4\theta}{3}\left(\frac{H^2}{4\pi(\rho+p)}\right) + 4\theta\left(\frac{H^2}{4\pi(2-\gamma)}\right)$$
$$- \frac{2H_{,0}^2}{4\pi(\rho+p)} - \frac{2H_{3,0}H_3}{4\pi\rho(2-\gamma)} + \frac{(1-\gamma)}{(\gamma+1)t}\left(\frac{H^2}{4\pi\rho(2-\gamma)}\right)$$

where, $\theta = -2/[(1+\gamma)t]$. The function T_2 depends on γ and t, since the magnetic field, the energy density and the pressure are all functions of γ and t. In other words, $T_2(\gamma,t)$ has the form

$$T_2(\gamma,t) = \frac{C(\gamma)}{t},$$

where,

$$C(\gamma) = \frac{45\gamma^4 - 36\gamma^3 - 126\gamma^2 + 212\gamma - 127}{3(\gamma-3)(\gamma-2)(\gamma+1)^2}.$$

We can easily show that $T_1(\gamma,t)$ is positive and $T_2(\gamma,t)$ is negative, for all the allowed values of $1/3 \le \gamma \le 1$.

References

1. Punsly, B.: Black Hole Gravitohydromagnetics. Springer, Berlin (2001)
2. Jackson, J.D.: Classical Electrodynamics. Wiley, New York (1975)
3. Carot, J., Tupper, B.O.J.: Phys. Rev. D **59**, 124017 (1999)
4. Dimmelmeier, H., Font, A.J., Müller, E.: Astron. Astrophys. **393**, 523 (2002)
5. Fryer, C.L., Holz, D.E., Hughes, S.T.: Astrophys. J. **565**, 430 (2002)
6. Baumgarte, T.W., Shapiro, S.L.: Astrophys. J. **585**, 930 (2003)
7. Kronberg, P.P.: Rep. Prog. Phys. **57**, 325 (1994)
8. Kronberg, P.P., Perry, J.J., Zukowski, E.L.H.: Astrophys. J. **387**, 528 (1992)
9. Wolfe, A.M., Lanzetta, K.M., Oren, A.L.: Astrophys. J. **388**, 17 (1992)
10. Hacyan, S.: Astrophys. J. **273**, 421 (1983)
11. Papadopoulos, D.B., Esposito, F.P.: Astrophys. J. **257**, 10 (1982)
12. Adams, J., Danielsson, U.H., Grasso, D., Rubinstein, H.: Phys. Lett. B **388**, 253 (1996)
13. Barrow, J.D., Ferreira, P.G., Silk, J.: Phys. Rev. D **78**, 3610 (1997)
14. Durrer, R., Kahniashvil, T., Yates, A.: Phys. Rev. D **58**, 123004 (1998)
15. Jedamzik, K., Katalinic, V., Olinto, A.: Phys. Rev. Lett. **85**, 700 (2000)
16. Papadopoulos, D.B., Vlahos, L., Esposito, F.P.: Astron. Astrophys. **382**, 1 (2002)
17. Weinberg, S.: Gravitation and Cosmology. Wiley, New York (1972)
18. Narlikar, J.V.: Introduction to Cosmology. Jones & Bartlett, Boston (1983)
19. Zel'dovich, Ya.B., Novikov, I.D.: The Structure and the Evolution of the Universe. University of Chicago Press, London (1983)
20. Chandrasekhar, S.: Hydrodynamic and Hydromagnetic Stability. Oxford Univ Press, Oxford (1961)
21. Ellis, G.F.R.: Lectures in general relativity and cosmology. In Sachs, R.K. (ed.) Proceedings of the International School of Physics Enrico Fermi XLVII (1971)
22. Hunter, C.: Astrophys. J. **139**, 570 (1964)
23. Hawking, S.W.: Astrophys. J. **145**, 544 (1966)
24. Thorne, K.S.: Astrophys. J. **148**, 51 (1967)
25. Jacobs, K.C.: Astrophys. J. **155**, 379 (1969)
26. Cowling, T.G.: Mon. Not. Roy. Astron. Soc. **101**, 367 (1941)
27. Parker, E.N.: Astrophys. J. **145**, 811 (1966)
28. Krall, N.A., Trivielpiece, A.W.: Principles of Plasma Physics. McGraw–Hill, New York (1973)
29. Jackson, J.: Proc. Roy. Soc. Lond. A **200**, 248 (1972)
30. Durrer, R., Ferreira, P.G., Kahniashvil, T.: Phys. Rev. D **61**, 43001 (2001)
31. Vlahos, L., Tsagas, C.G., Papadopoulos, D.B.: Astrophys. J. Lett. **629**, L9 (2005)

32. Tsagas, C.G., Maartens, R.: Phys. Rev. D **61**, 83519 (2000)
33. Parker, E.N.: Astrophys. J. **414**, 389 (1993)
34. Galeev, A.A., Sagdev, R.Z.: In: Galeev A.A., Sudan R.N. (eds.) Basic Plasma Physics, vol 2. North Holland, Amsterdam (1984)
35. Hamberger, S.M., Friedmann, M.: Phys. Rev. Lett. **21**, 674 (1968)
36. Kronberg, P.P., Colgate, S.A., Li, H., Dufton, Q.W.: Astrophys. J. **604**, L77 (2004)
37. Brandenburg, A., Enqvist, K., Olesen, P.: Phys. Rev. D **54**, 1291 (1996)